Let us apply the Feynman diagram methed to CS-p.f.:

$$Z_k(M) = \int \exp\left(2\pi\sqrt{-1}\, k\, CS(A)\right) \mathcal{D}A$$

$\longrightarrow$ degenerates along gauge orbit $\mathcal{G}_\alpha$

finite-dim. analog:

Let G be an l-dim. Lie group $\circlearrowright \mathbb{R}^n$

suppose $f(x_1, \dots, x_n) = Q(x_1, \dots, x_n) + \sum_{ijk} \lambda_{ijk} x^i x^j x^k$

is invariant under $\overset{G}{\circlearrowright}$.

Take $F: \mathbb{R}^n \longrightarrow \mathbb{R}^l$ be smooth function with one zero along each G-orbit:

$$\vec{x} \in \mathbb{R}^n : F(\vec{x}) = 0 \longrightarrow \text{orbit } G\vec{x}$$

$$\text{set } \varphi: G \longrightarrow \mathbb{R}^l, \ \varphi(g) = F(g\vec{x})$$

$$J(\vec{x}) = D_e \varphi \quad \text{Jacobian}$$

$$\longrightarrow \operatorname{Det} J(\vec{x}) = \frac{vol(G\vec{x})}{vol(G)}$$

Then
$$Z_k = \int_{\mathbb{R}^n} e^{\sqrt{-1}kf(x_1,\dots,x_n)} dx_1 \cdots dx_n$$

$$\longrightarrow \int_{\mathbb{R}^n} e^{\sqrt{-1}kf(x_1,\dots,x_n)} \delta(F(\vec{x})) \det J(\vec{x}) dx_1 \cdots dx_n \quad (*)$$

The Dirac δ-distribution can be rep. as

$$\delta(F(\vec{x})) = \frac{1}{(2\pi)^\ell} \int_{\mathbb{R}^\ell} e^{\sqrt{-1}\sum_j F_j(\vec{x})\phi_j} d\phi_1 \cdots d\phi_\ell$$

and (*) becomes

$$\frac{1}{(2\pi)^\ell} \int_{\mathbb{R}^{n+\ell}} e^{\sqrt{-1}\left(\kappa f(x_1,\dots,x_n) + \sum_j F_j(\vec{x})\phi_j\right)} \times \det J(\vec{x})\, dx_1 \cdots dx_n\, d\phi_1 \cdots d\phi_\ell$$

Now use

$$\det J(\vec{x}) = \int e^{\sqrt{-1}\sum_{ij} \bar{c}_i J_{ij} c_j} d\bar{c}_1 \cdots d\bar{c}_\ell\, dc_1 \cdots dc_\ell$$

where $\{c_i\}$, $\{\bar{c}_i\}$, $1 \le i \le \ell$ are Grassman variables satisfying

$$c_i c_j + c_j c_i = 0, \quad \bar{c}_i \bar{c}_j + \bar{c}_j \bar{c}_i = 0, \quad c_i \bar{c}_j = \bar{c}_j c_i,$$
$$1 \le i, j \le \ell.$$

In total we (*) $\longrightarrow$

$$\frac{1}{(2\pi)^\ell} \int_{\mathbb{R}^{n+\ell}} e^{\sqrt{-1}\left(\kappa f(x_1,\dots,x_n) + \sum_j F_j(\vec{x})\phi_j + \sum_{ij} \bar{c}_i J_{ij} c_j\right)} d\{x_i\}\, d\{\phi_i\}\, d\{c_i\}$$

In the case of CS-theory, restrict to

$$d_\alpha^* A = 0 \quad (\text{analog of } F(\vec{x}) = 0)$$

"gauge fixing"

Let M be a closed oriented 3-mfd
Fix α ($su(2)$ connection) such that

$$H^*(M, \mathfrak{g}_\alpha) = 0$$

Let $\{I_a\}$, $a=1,2,3$ be an orthonormal basis of the Lie algebra $\mathfrak{g} = su(2)$
Let $L \in \Omega^2(M \times M; \mathfrak{g} \otimes \mathfrak{g})$ and write

$$L = \sum_{a,b} L_{ab}(x,y)\, I_a \wedge I_b$$

$\rightarrow$ for L to be Green form, it must satisfy

$$\underset{\substack{\uparrow \\ \text{covariant der.}}}{d_{(\alpha,\alpha)}} L_{(a,b)}(x,y) = -\delta_{ab}\, \delta(x,y)$$

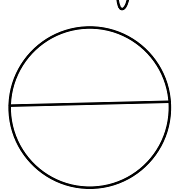
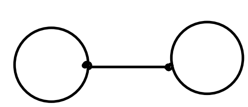
where $\delta(x,y)$, $(x,y) \in M \times M$ satisfies

$$\int_{M \times M} \delta(x,y)\, \psi(x,y) = \int_M \psi(x,x)$$

for a 3-form $\psi(x,y)$ on $M \times M$. We have

$$L_{ab}(x,y) = -L_{ba}(y,x)$$

Denote by $\{\gamma_{abc}\}$ the structure constants of $\mathfrak{g}$
Then Feynman diagrams

$T_1 =$  and $T_2 =$ 

become

$$I_{T_1}(M,\alpha) = -\frac{1}{8}\sum \int_{M\times M\setminus\Delta} \gamma_{a_1b_1c_1}\gamma_{a_2b_2c_2}$$

$$\times L_{a_1c_1}(x_1,x_1)\wedge L_{a_2c_2}(x_2,x_2)\wedge L_{b_1b_2}(x_1,x_2)$$

$$I_{T_2}(M,\alpha) = \frac{1}{12}\sum \int_{M\times M\setminus\Delta} \gamma_{a_1b_1c_1}\gamma_{a_2b_2c_2}$$

$$\times L_{a_1a_2}(x_1,x_2)\wedge L_{c_1c_2}(x_1,x_2)\wedge L_{b_1b_2}(x_1,x_2)$$

Fix framing s of M (trivialization of tangent bundle) $\longrightarrow CS_{grav}(g,s)$

gravitational CS-invariant

Theorem 1:

Let M be a closed oriented 3-mfd

Then for an $SU(2)$ flat connection α,

$$I_{T_1}(M,\alpha) + I_{T_2}(M,\alpha) - \frac{1}{4\pi} CS_{grav}(g,s)$$

is a top. invariant of M and does not depend on the choice of a Riemannian metric for M.

Sketch of proof:

Consider a one-parameter family of Riemannian metrics $\{g^t\}$

$\longrightarrow$ Green forms L^t

Denote by ∇ diff. operator with respect to t

Since $d_{(\alpha,\alpha)} L^t$ = t indep.

$$\longrightarrow \quad d_{(\alpha,\alpha)} \nabla L^t = 0$$

Since $H^*(M, \mathfrak{g}_\alpha) = 0$, we have

$\exists$ 1-form B_t on $M \times M$ with values in $\mathfrak{g} \otimes \mathfrak{g}$ satisfying

$$\nabla L^t = d_{(\alpha,\alpha)} B^t$$

Express B^t as

$$B^t = \sum B^t_{ab}(x,y)\, I_a \wedge I_b$$

Compactify the space $Conf_2(M) = M \times M \setminus \Delta$

$$\longrightarrow \quad \overline{Conf_2(M)}$$

Integrals I_{Γ_1} and I_{Γ_2} are over $\overline{Conf_2(M)}$

One then computes using Stokes' th.

$$\nabla I_{\Gamma_1}$$

$$= \frac{1}{8} \int_{\overline{Conf_2(M)}} \sum \gamma_{a_1 b_1 c_1} \gamma_{a_2 b_2 c_2} B^t_{a_1 c_1}(x_1, x_1) L^t_{a_2 c_2}(x_2, x_2)\, \delta_{b_1 b_2} \delta(x_1 x_2)$$

$$+ \frac{1}{8} \int_{\overline{Conf_2(M)}} \sum \gamma_{a_1 b_1 c_1} \gamma_{a_2 b_2 c_2} L^t_{a_1 c_1}(x_1, x_1) B^t_{a_2 c_2}(x_2, x_2)\, \delta_{b_1 b_2} \delta(x_1, x_2)$$

$$+ \int_{\partial \overline{Conf_2(M)}} \cdots$$

$$= \frac{1}{4} \int_M \sum \gamma_{acd} \gamma_{aef} B^t_{cd}(x, x) L^t_{ef}(x, x) + \int_{\partial \overline{Conf_2(M)}} \cdots \qquad (1)$$

Similarly,

$$\nabla I_{\Gamma_2}$$

$$= \frac{1}{4} \int_M \sum (\gamma_{ace} \gamma_{afd} - \gamma_{acf} \gamma_{aed}) B^t_{cd}(x, x) L^t_{ef}(x, x)$$

$$+ \int_{\partial \overline{Conf_2(M)}} \cdots \qquad (2)$$

Then $\nabla I_{\Gamma_1} + \nabla I_{\Gamma_2} = 0$ follows from Jacobi-identity and $\int_{\partial \overline{Conf_2(M)}} \cdots$ is cancelled against variation of CS_{grav}. $\square$